Optimization
B.Math. (Hons.) IInd year
First Semestral exam 2015
Instructor : B.Sury
Answer SIX questions INCLUDING question 8.
Question 8 carries 10 marks; the others carry 8 marks each.

Q 1.
(i) If $A, B, A+B$ are invertible square matrices, then obtain the inverse of $A^{-1}+B^{-1}$ in terms of $A, B, A+B$.
(ii) If $A \in M_{m,n}(\mathbf{R}), B \in M_{n,m}(\mathbf{R})$, and if $I_m - AB$ is invertible, prove that $I_n - BA$ is invertible.

Q 2. Consider $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 4 & 2 \end{pmatrix}$. Find the nullity of A, a set of rows which forms a basis of the row space and a set of rows which forms a basis of the column space.

Q 3.
(i) Prove that for any $A \in M_{m,n}(\mathbf{C})$, any consistent system $Ax = b$ possesses a unique minimum norm solution.
(ii) If $v_1 = (1,-1,0,0), v_2 = (0,1,-1,0), v_3 = (0,0,1,-1)$, find an orthonormal basis for the subspace of $\mathbf{R}^4$ spanned by v_1, v_2, v_3.

Q 4.
(i) Let A and B be rectangular, complex matrices with the same number of rows. If the column space of B is a subspace of the column space of A, show that $B = AC$ for some matrix C.
(ii) For any real matrix A, prove that the rank of A equals that of AA^t and of A^tA.

Q 5. Prove that an $n \times m$ matrix G is a generalized inverse of an $m \times n$ matrix A if, and only if, $AGA = A$.

Q 6. Determine a singular value decomposition of $\begin{pmatrix} 2 & 1 & 0 & 1 \\ -1 & 1 & 1 & -2 \\ 1 & 2 & 1 & -1 \end{pmatrix}$.

Q 7. Prove that a matrix $P \in M_n(\mathbf{C})$ is an orthogonal projector if, and only if, $P^*P = P$.

Q 8. Let $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$. Then, use the simplex method to determine a tuple (x_1, x_2, x_3) such that $f(x) := x_1 + 2x_2 + 3x_3$ is maximized subject to the constraints $Ax \leq \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $x_i \geq 0$ for $i = 1, 2, 3$. Further, determine a tuple (y_1, y_2, y_3) such that $f(y)$ is minimized subject to $Ay \geq \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.